PSI Center for Photon Science

EPFL

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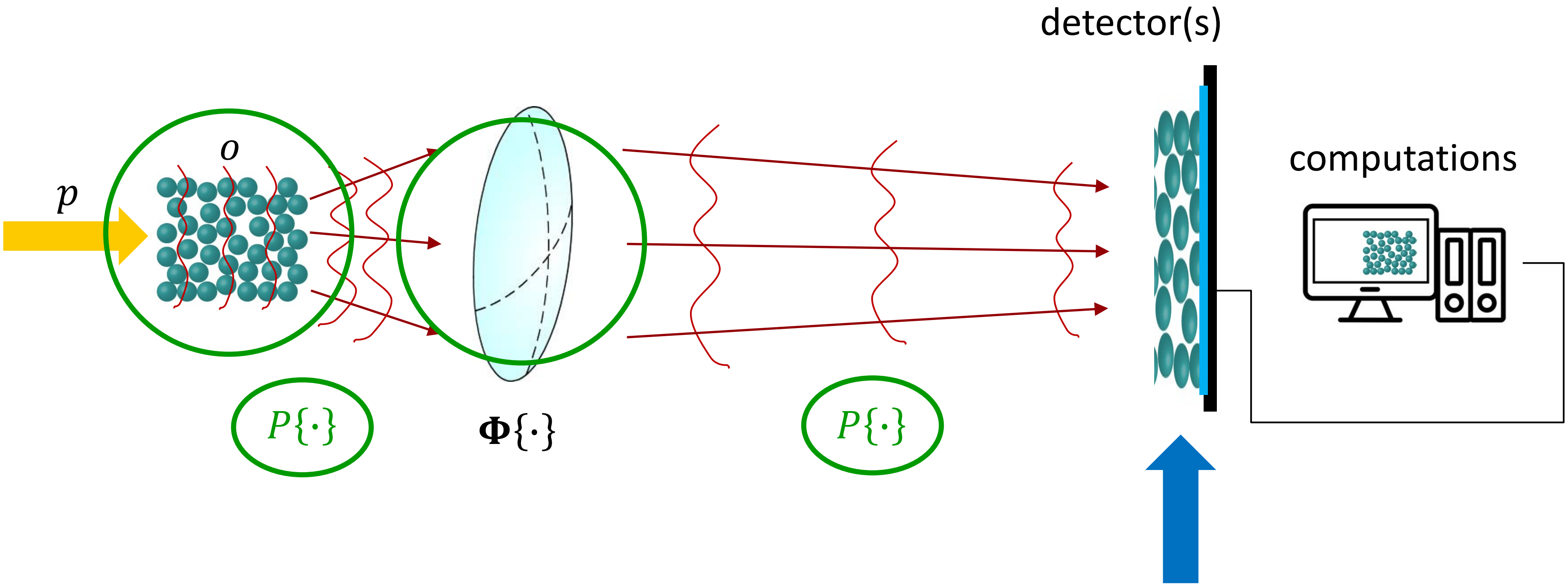
Chapter 3 – Detector sampling and DFT

PHYS-715 Physical Optics and Advanced Imaging 2025

CXI

Lenses and imaging

A more detailed model



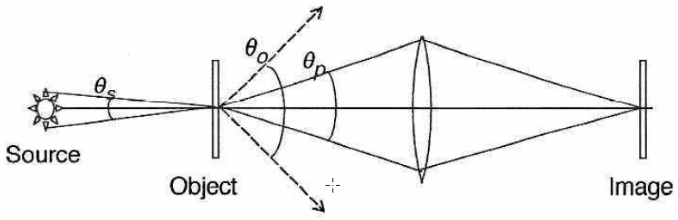
Recap on imaging

Coherent vs incoherent imaging

Coherent

$$I_i(u,v) = \left| \iint U_g(\xi,\eta)\, h(u-\xi, v-\eta)\, d\xi d\eta \right|^2$$

$$U_i(x,y) = \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{ U_g(x,y) \right\} P\left(-\lambda z_i f_x, -\lambda z_i f_y\right) \right\}$$



Coherent if $\theta_s << \theta_p$

Incoherent if $\theta_s \geq \theta_o + \theta_p$

Partially coherent in-between

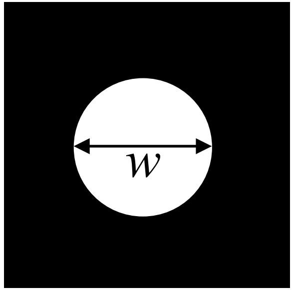
Incoherent

$$I_i(u,v) = \kappa \iint I_g(\xi,\eta) \left| h(u-\xi, v-\eta) \right|^2 d\xi d\eta$$

$$I_i(u,v) = \kappa \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{ I_g(\xi,\eta) \right\} OTF\left(f_x, f_y\right) \right\}$$

$$OTF\left(f_x, f_y\right) = P\left(-\lambda z_i f_x, -\lambda z_i f_y\right) \bigstar P\left(-\lambda z_i f_x, -\lambda z_i f_y\right)$$

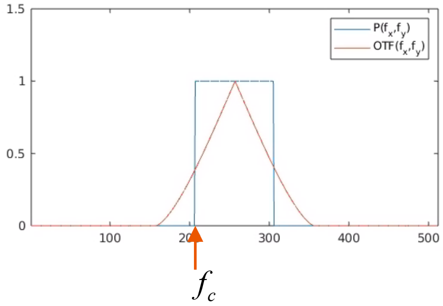
Circular pupil



Exit / entrance pupil diameter wi

$$P(x,y) = \mathrm{circ}\left(\frac{r}{w_i/2}\right)$$

$$H(f_x,f_y) = \mathrm{circ}\left(\frac{\lambda z_i f_r}{w_i/2}\right)$$

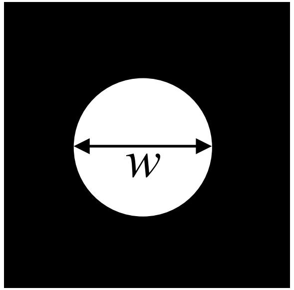


$$f_c = \frac{1}{2}\frac{w_i}{\lambda z_i}$$

Coherent cutoff frequency

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Circular pupil

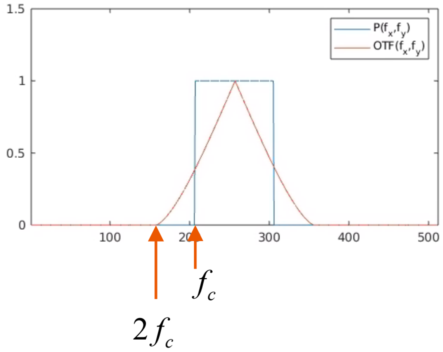


Exit / entrance pupil diameter wi

$$P(x,y) = \mathrm{circ}\left(\frac{r}{w_i/2}\right)$$

$$H(f_x,f_y) = \mathrm{circ}\left(\frac{\lambda z_i f_r}{w_i/2}\right)$$

$$OTF\left(f_x,f_y\right) = H(f_x,f_y) \star H(f_x,f_y)$$

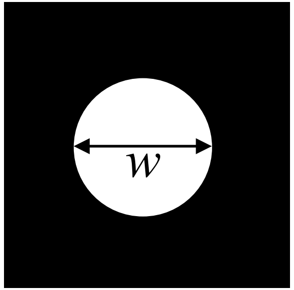


$$f_c = \frac{1}{2}\frac{w_i}{\lambda z_i}$$ Coherent cutoff frequency

$$f_i = \frac{w_i}{\lambda z_i}$$ Incoherent cutoff frequency

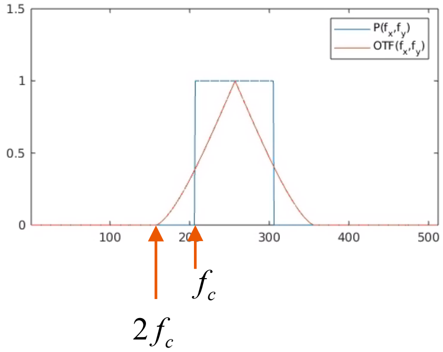
J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Circular pupil – non paraxial



Pupil diameter w

$$P(x,y) = \mathrm{circ}\left(\frac{r}{w_i/2}\right)$$



$$f_c = \left(\frac{w_i}{2z_i}\right)\frac{1}{\lambda} = \frac{\mathrm{NA}}{\lambda}$$

Coherent cutoff frequency

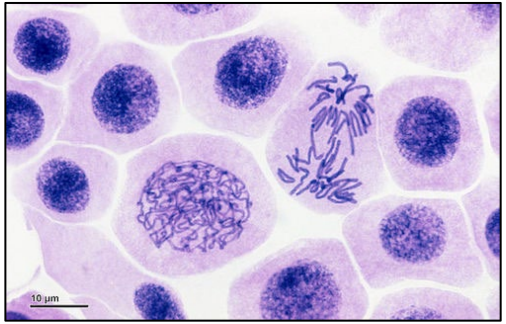
$$f_i = \frac{w_i}{\lambda z_i} f_c = \left(\frac{w_i}{2z_i}\right)\frac{2}{\lambda} = \frac{2\mathrm{NA}}{\lambda}$$

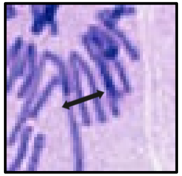
Incoherent cutoff frequency

J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

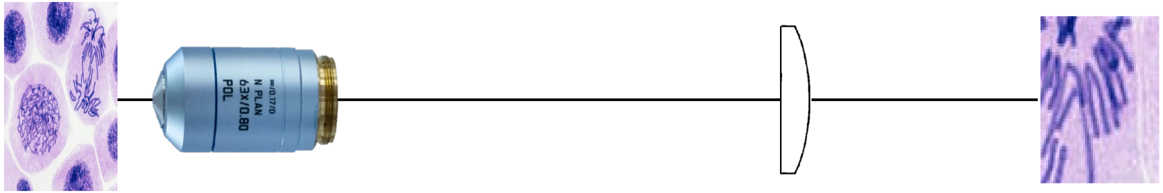
An optical microscope

What NA do I need to resolve these DNA strains?





3 µm



How small should my pixel size be to ensure no info is lost?

Onion root cells viewed through a light microscope. The DNA is stained purple, which allows us to see cells in various stages of cell division. Image credit: “Mitosis” by Josef Reischig, CC BY-SA 3.0.
https://www.khanacademy.org/science/hs-bio/x230b3ff252126bb6:from-cells-to-organisms/x230b3ff252126bb6:the-cellular-basis-of-life/a/cell-biology-and-microscopy

Course contents

Cover the unifying concepts of imaging, including with electrons, optics, X-rays.
In 2D, 3D, and beyond.

Chapter 3 – Detector, sampling, and DFT

- The Whittaker-Shannon sampling theorem
- Nyquist sampling
- Aliasing
- Discrete Fourier transform (DFT)

The Whittaker-Shannon sampling theorem

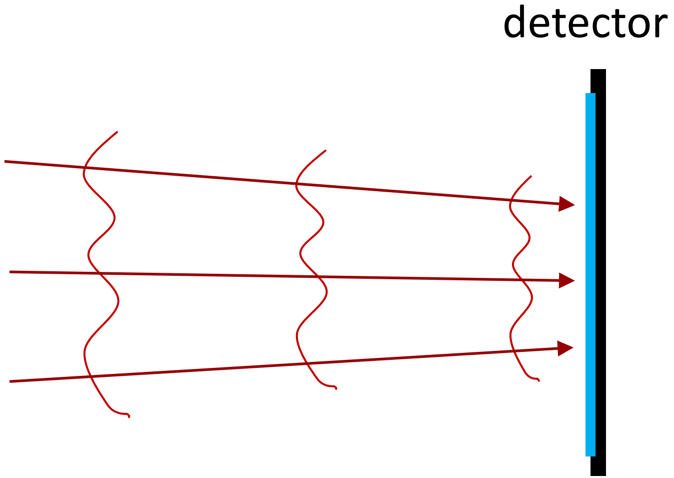
The Whittaker-Shannon sampling theorem

Sampling → Very general analog to digital transformation

- Sampling for sound
- Measuring images with pixelated cameras
- Measuring any type of intensities with detectors
- Simulating masks
- Simulation of propagation

The Whittaker-Shannon sampling theorem

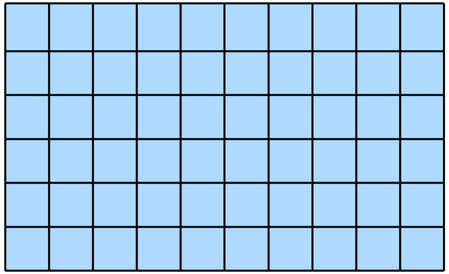
Intensity of the field is measured by the detector



$$g(x, y) = |\psi(x, y)|^2$$

- Image
- Propagated wave

Assuming a detector with perfectly sharp response



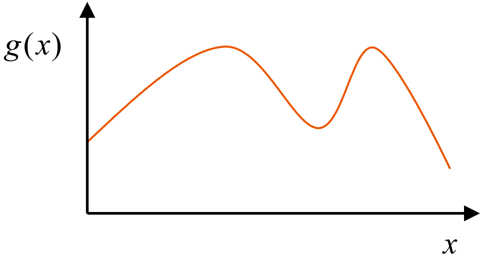
$$g_{m,n} = g(m\Delta x, n\Delta y)$$

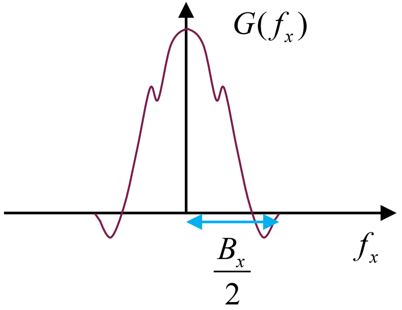
$\Delta x, \Delta y$ Pixel periods (pixel separation)

m, n Integer indices

Band-limited function

$$\mathcal{F}\{g(x)\} = G(f_x)$$

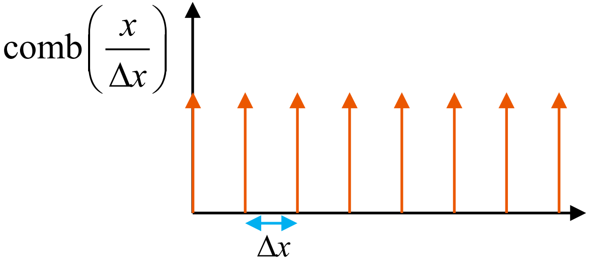




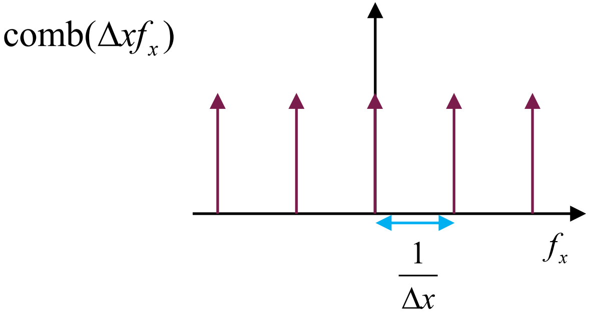
Not that the full bandwidth of the signal $B_x = 2f_{max}$

comb function

$$\mathrm{comb}\left(\frac{x}{\Delta x}\right)=\sum_{m=-\infty}^{\infty}\delta\left(\frac{x}{\Delta x}-m\right)=\Delta x\sum_{m=-\infty}^{\infty}\delta\left(x-m\Delta x\right)$$



$$\mathcal{F}\left\{\mathrm{comb}\left(\frac{x}{\Delta x}\right)\right\}=\Delta x\,\mathrm{comb}\left(\Delta x f_x\right)$$

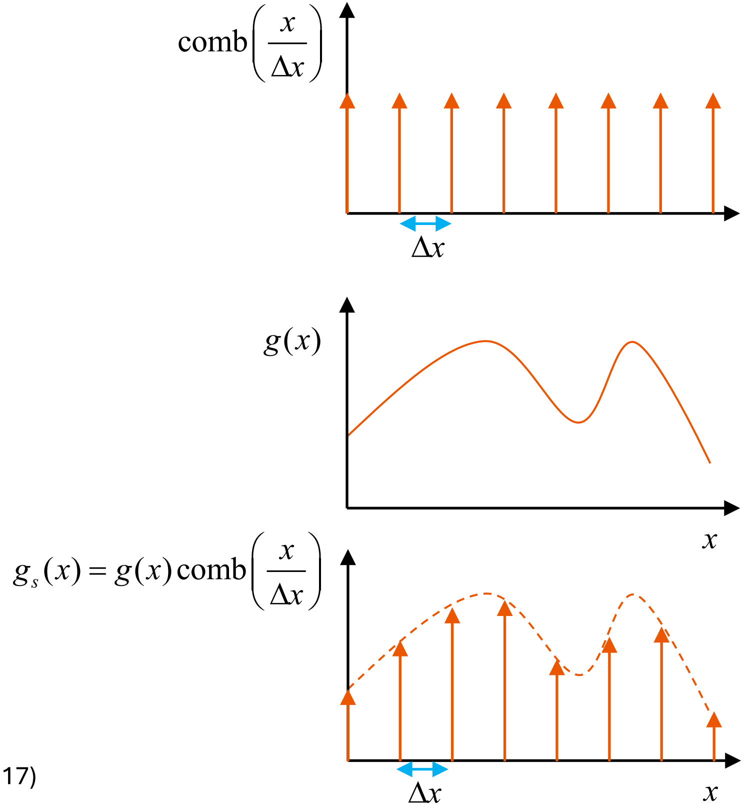


The Whittaker-Shannon sampling theorem

$$\operatorname{comb}\left(\frac{x}{\Delta x}\right) = \sum_{m=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - m\right) = \Delta x \sum_{m=-\infty}^{\infty} \delta\left(x - m\Delta x\right)$$

Sampled function in continuous space

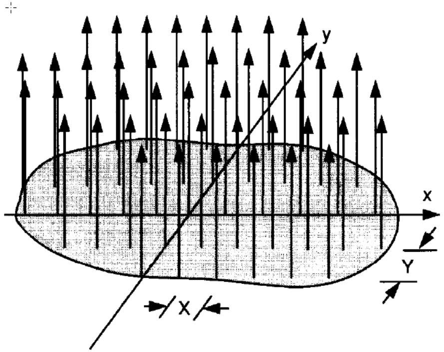
$$g_s(x) = g(x)\operatorname{comb}\left(\frac{x}{\Delta x}\right) = \sum_{m=-\infty}^{\infty} g\left(m\Delta x\right)\delta\left(\frac{x}{\Delta x} - m\right)$$



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

The Whittaker-Shannon sampling theorem

$$\operatorname{comb}\left(\frac{x}{\Delta x}\right)\operatorname{comb}\left(\frac{y}{\Delta y}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta\left(\frac{x}{\Delta x}-m,\frac{y}{\Delta y}-n\right)$$



Sampled function in continuous space

$$g_s(x,y)=g(x,y)\operatorname{comb}\left(\frac{x}{\Delta x}\right)\operatorname{comb}\left(\frac{y}{\Delta y}\right)=\sum_{m=-\infty}^{\infty}g\left(m\Delta x,n\Delta y\right)\delta\left(x-m\Delta x,y-n\Delta y\right)$$

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The Whittaker-Shannon sampling theorem

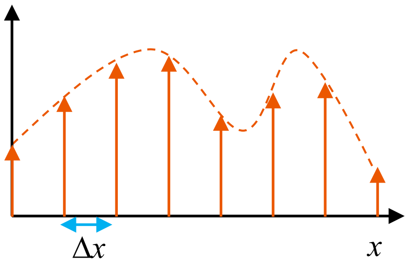
$$g_s(x,y) = g(x,y)\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)\mathrm{comb}\left(\frac{y}{\Delta y}\right)$$

$$G_s\left(f_x,f_y\right) = G\left(f_x,f_y\right) * \left[\Delta x \Delta y\,\mathrm{comb}\left(\Delta x f_x\right)\mathrm{comb}\left(\Delta y f_y\right)\right]$$

$$G_s\left(f_x,f_y\right) = G\left(f_x,f_y\right) * \left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\delta\left(f_x - \frac{p}{\Delta x}, f_y - \frac{q}{\Delta y}\right)\right]$$

$$G_s\left(f_x,f_y\right) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} G\left(f_x - \frac{p}{\Delta x}, f_y - \frac{q}{\Delta y}\right)$$

$$g_s(x) = g(x)\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)$$



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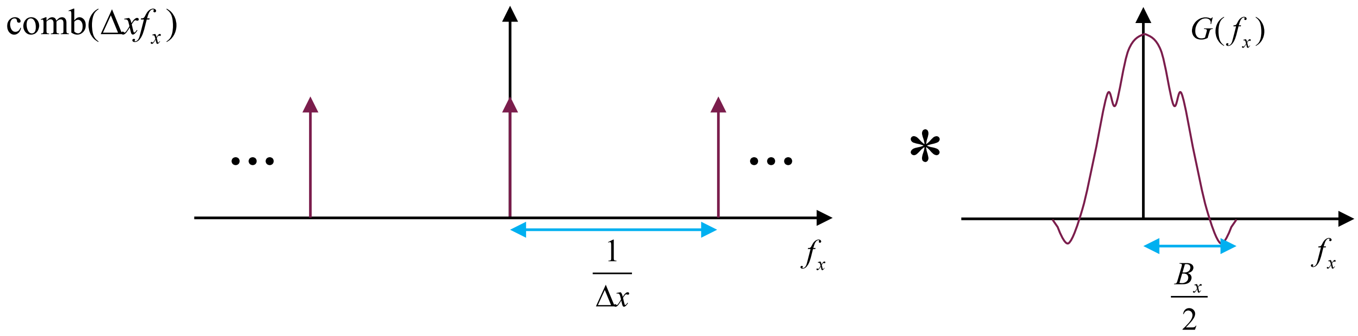
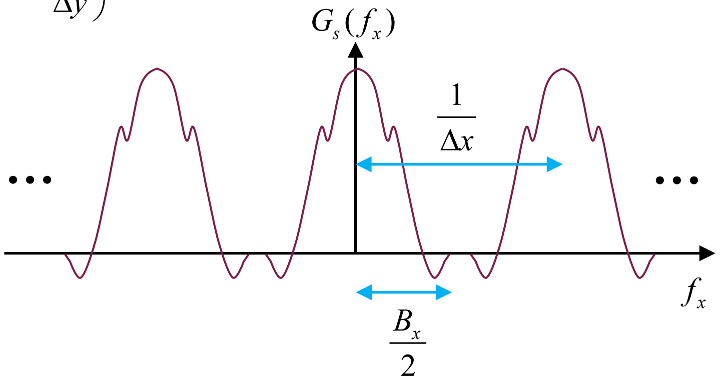
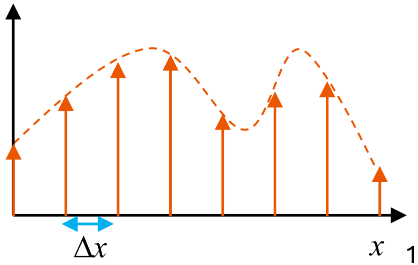
The Whittaker-Shannon sampling theorem



$$G_s\left(f_x, f_y\right) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} G\left(f_x - \frac{p}{\Delta x}, f_y - \frac{q}{\Delta y}\right)$$



$$g_s(x) = g(x)\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)$$



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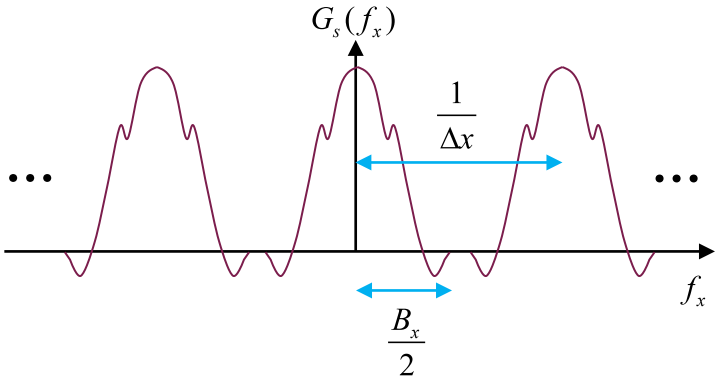
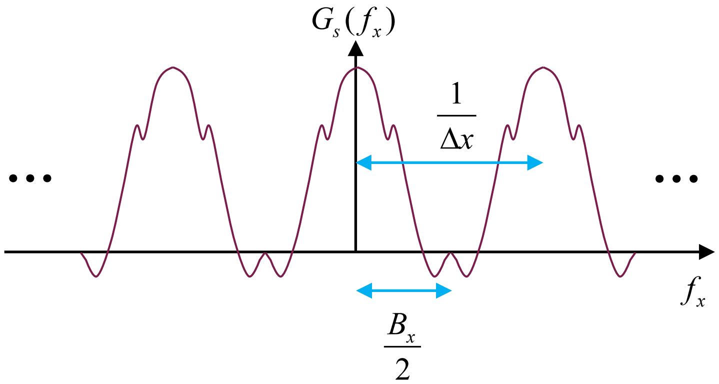
Nyquist sampling

Oversampled $\Delta x < \frac{1}{B_x}$



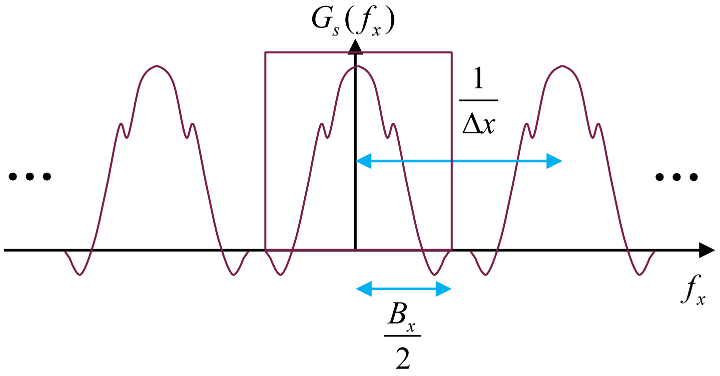
Nyquist sampled

$$\Delta x = \frac{1}{B_x} = \frac{1}{2 f_{max}}$$



J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Recovering the initial signal



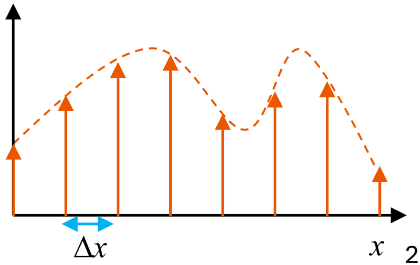
$$G(f_x, f_y) = G_s(f_x, f_y)\,\mathrm{rect}\left(\frac{f_x}{B_x}\right)\mathrm{rect}\left(\frac{f_x}{B_x}\right)$$

$$g(x,y) = g_s(x,y) * \left[B_x B_y \,\mathrm{sinc}(B_x x)\,\mathrm{sinc}(B_y y)\right]$$

$$g(x,y) = \left[\Delta x \Delta y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(m\Delta x, n\Delta y)\,\delta(x - m\Delta x, y - n\Delta y)\right] * \left[B_x B_y \,\mathrm{sinc}(B_x x)\,\mathrm{sinc}(B_y y)\right]$$

$$g(x,y) = \Delta x \Delta y B_x B_y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(m\Delta x, n\Delta y)\,\mathrm{sinc}(B_x[x - m\Delta x])\,\mathrm{sinc}(B_y[y - n\Delta y])$$

$$g_s(x) = g(x)\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)$$

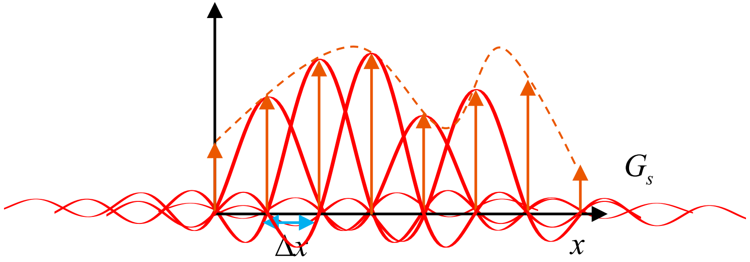


J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Recovering the initial signal

Sinc interpolation → Band-limited signals

$$g(x,y) = \Delta x \Delta y B_x B_y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g(m\Delta x, n\Delta y)\,\mathrm{sinc}\left(B_x\left[x - m\Delta x\right]\right)\mathrm{sinc}\left(B_y\left[y - n\Delta y\right]\right)$$

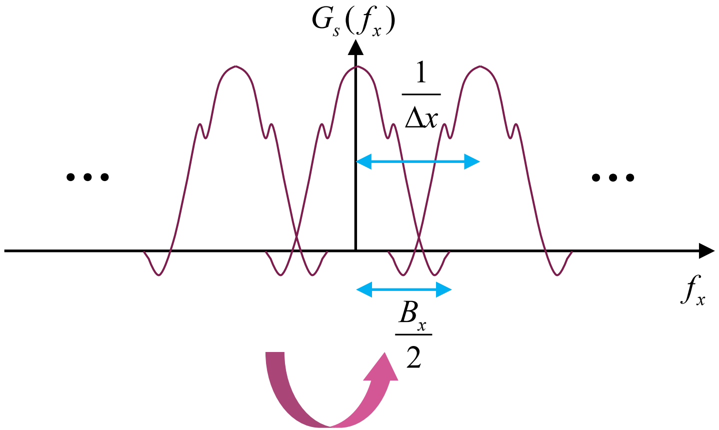


J. W. Goodman, Introduction to Fourier Optics, fourth edition. McMillan learning (2017)

Aliasing

Undersampled

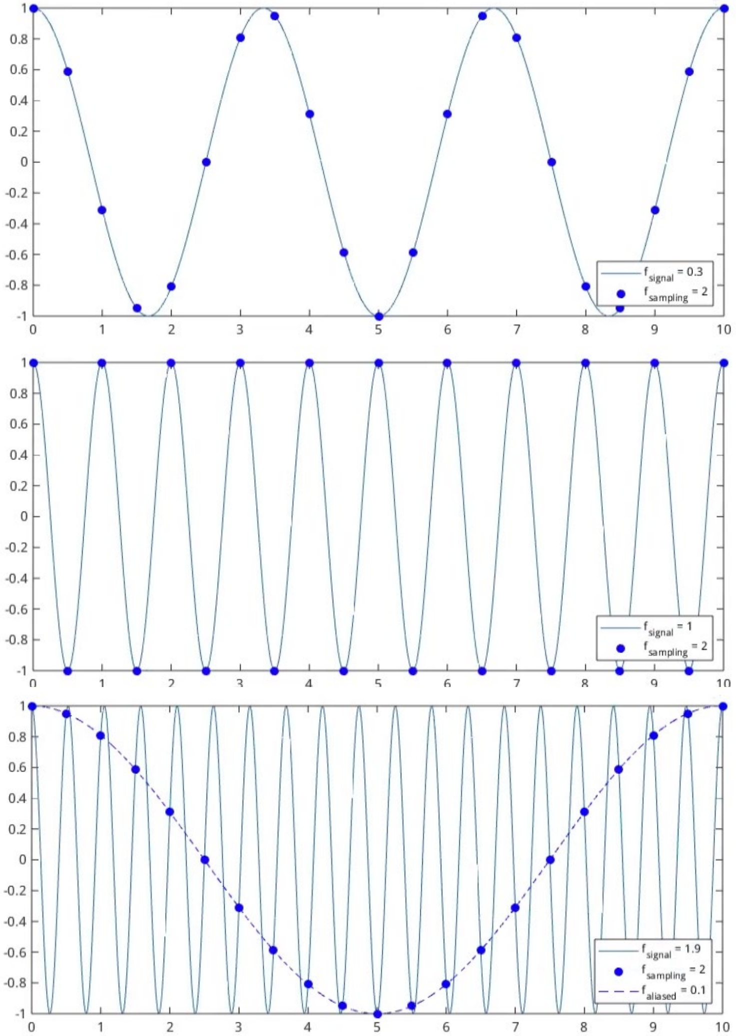
$$\Delta x > \frac{1}{B_x}$$



Large spatial frequencies start to “look like” smaller spatial frequencies

aliasing.m

Aliasing



aliasing.m

Aliasing in 2D

Edge smoothing is important when simulating pinholes or apertures

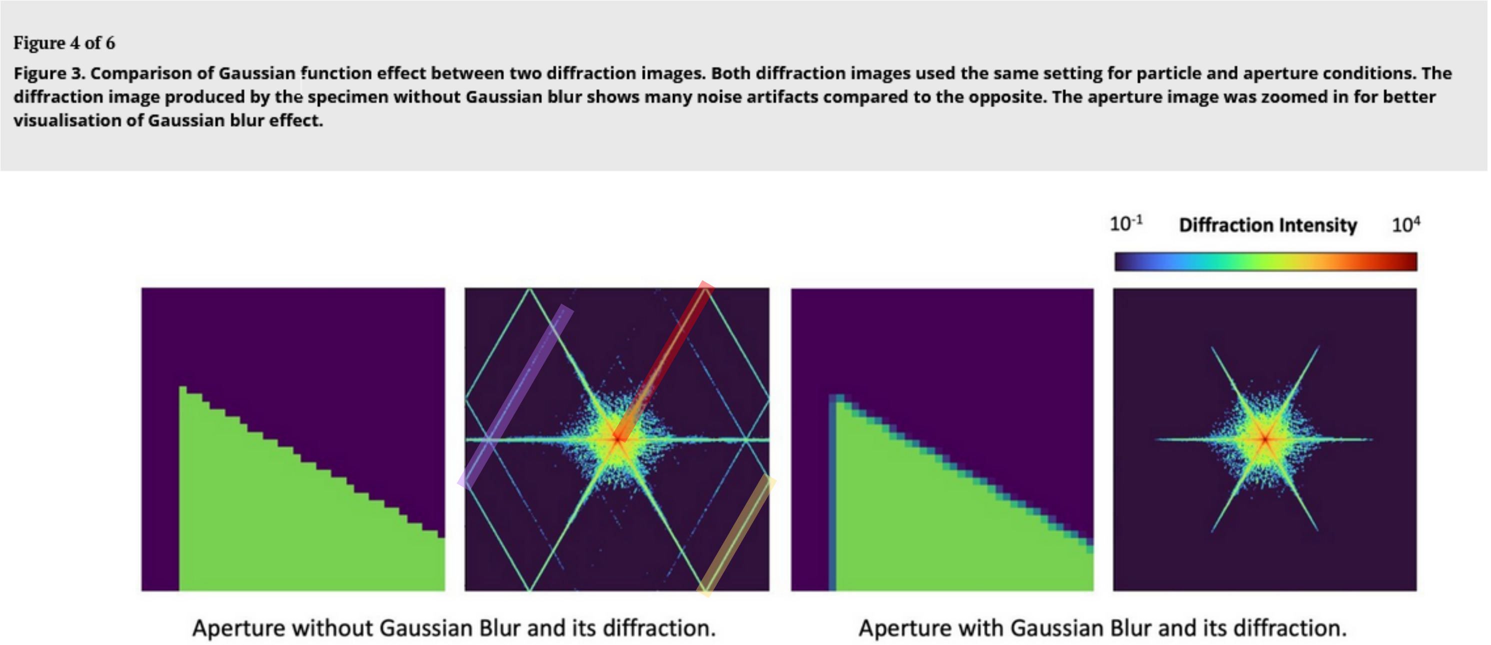


Figure 4 of 6

Figure 3. Comparison of Gaussian function effect between two diffraction images. Both diffraction images used the same setting for particle and aperture conditions. The diffraction image produced by the specimen without Gaussian blur shows many noise artifacts compared to the opposite. The aperture image was zoomed in for better visualisation of Gaussian blur effect.

Aperture without Gaussian Blur and its diffraction.

Aperture with Gaussian Blur and its diffraction.

https://www.tandfonline.com/doi/full/10.1080/00268976.2024.2388303#d1e1239

Aliasing in 2D

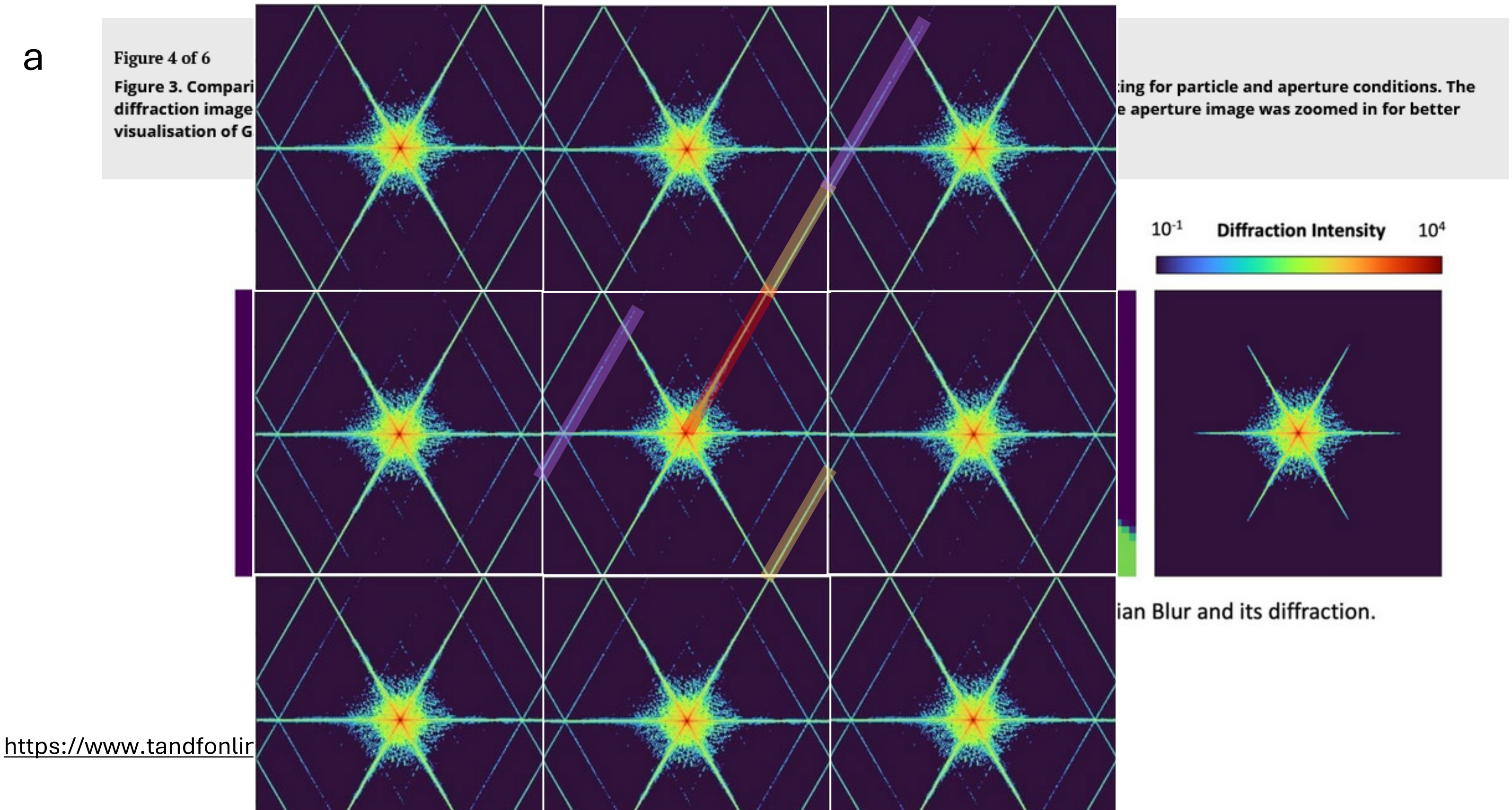


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Figure 4 of 6

Figure 3. Compari
diffraction image
visualisation of G

ting for particle and aperture conditions. The
e aperture image was zoomed in for better

ian Blur and its diffraction.

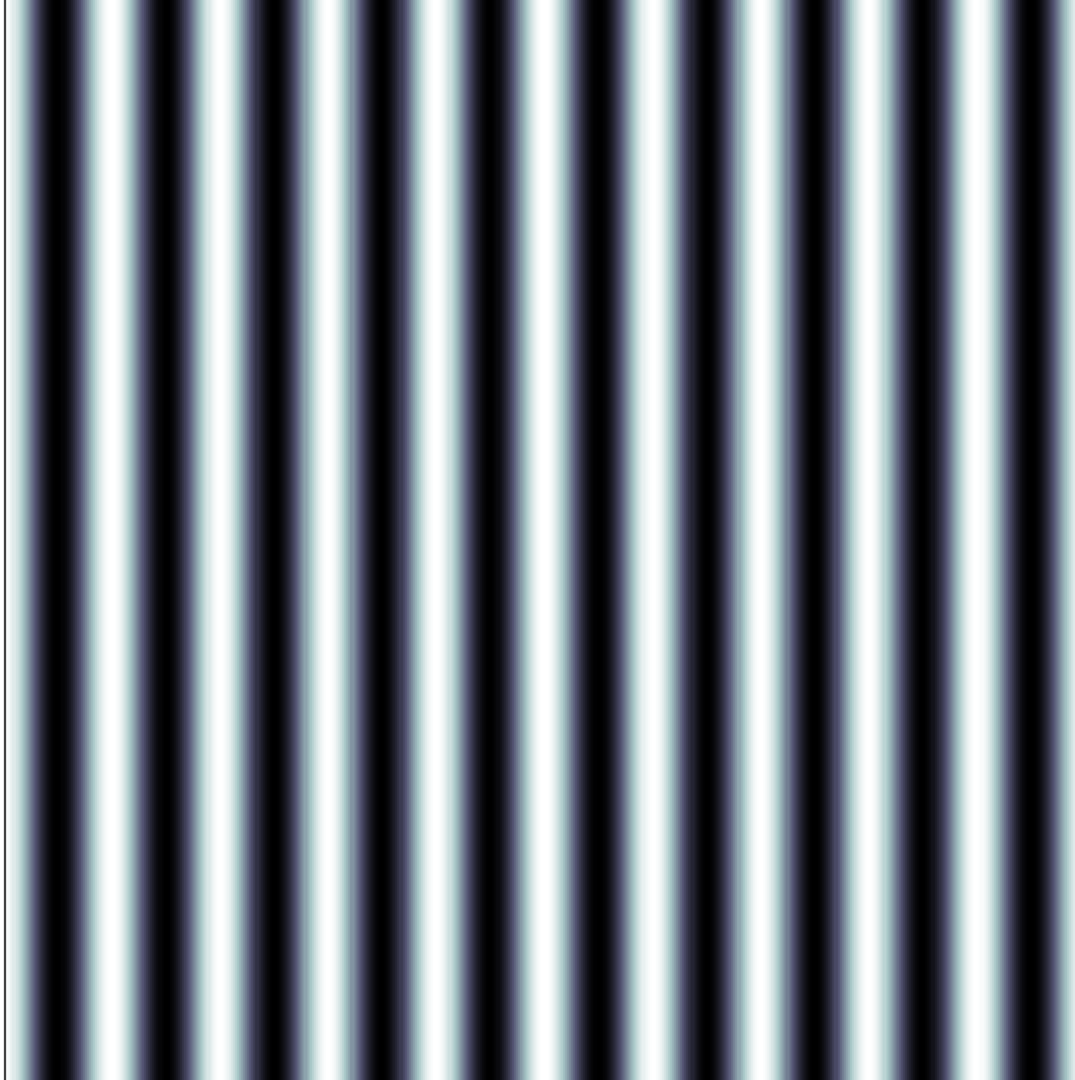
https://www.tandfonlir

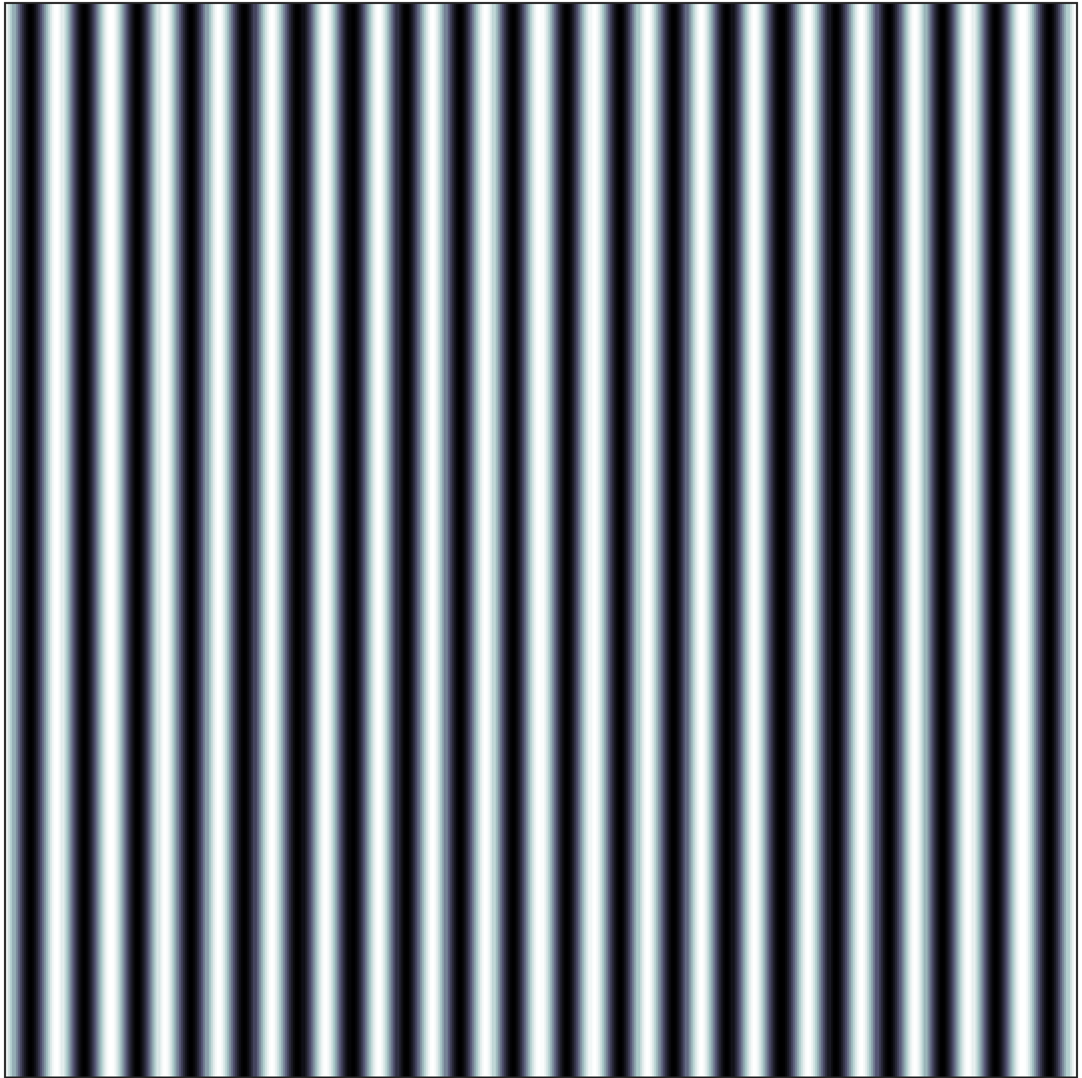
Aliasing in 2D

Subtle effect to notice by eye

One case where it is noticeable is if you have a periodic patterns

Image from https://graphicdesign.stackexchange.com/questions/125041/how-to-approach-cleaning-up-an-emergent-moire-pattern-in-a-low-resolution-image
https://play.google.com/store/apps/details?id=net.rahorton.fft&hl=en





Sampling for an imaging system (incoherent, paraxial)

Exit pupil diameter w. Distance to image z_i

$$P(x,y)=\operatorname{circ}\left(\frac{r}{w/2}\right)$$

$$H(x,y)=\operatorname{circ}\left(\frac{\lambda z_i f_r}{w/2}\right)$$

Coherent transfer function

$$f_c=\frac{w}{\lambda z_i}$$

OTF cutoff frequency (w is exit pupil diameter)

$$\Delta x\le\frac{1}{2f_c}=\frac{1}{2}\frac{\lambda z_i}{w}$$

Nyquist sampling detector spacing

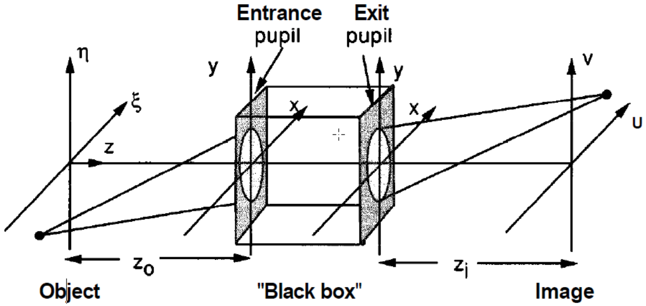
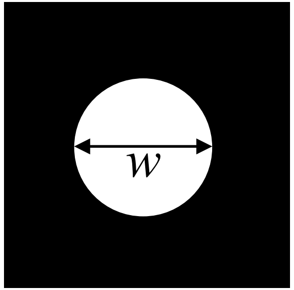
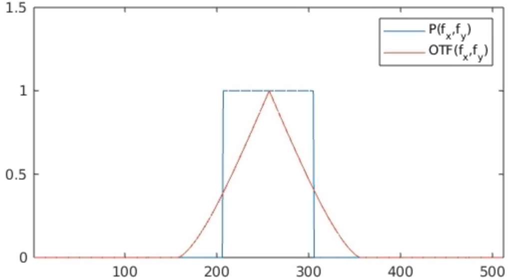


FIGURE 6.1
Generalized model of an imaging system.





Sampling for an imaging system (incoherent, non-paraxial)

Exit pupil diameter w. Distance to image z_i

$$\Delta x \leq \frac{1}{2f_c}$$

$$\Delta x \leq \frac{\lambda}{4\mathrm{NA}_X}$$

$$\Delta x \leq \frac{M\lambda}{4\mathrm{NA}_E}$$

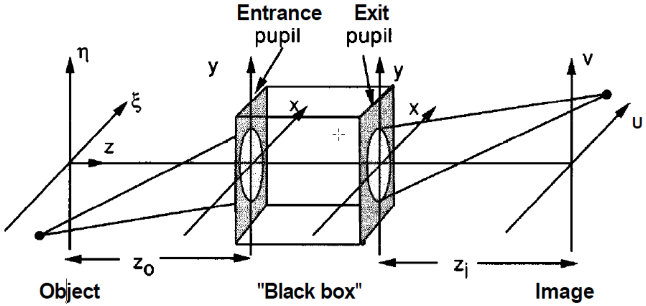
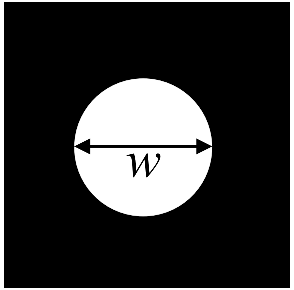
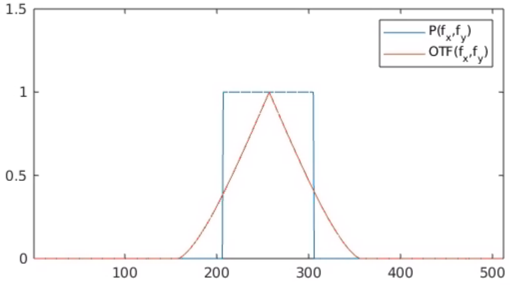


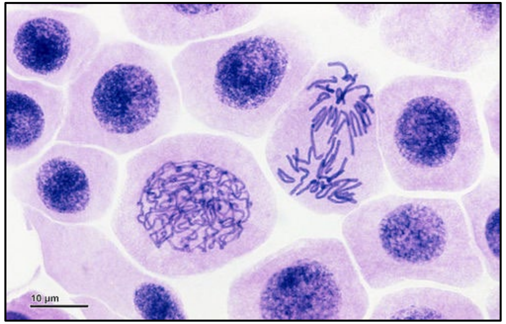
FIGURE 6.1
Generalized model of an imaging system.

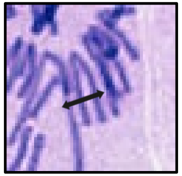




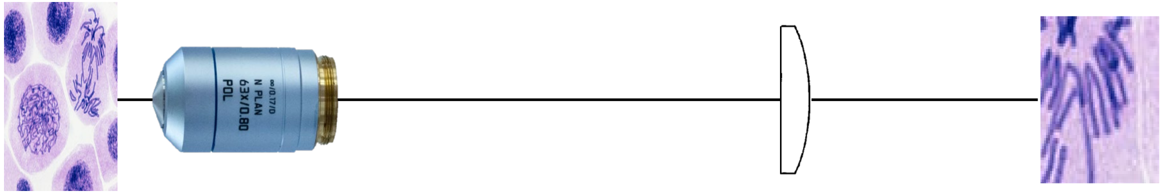
An optical microscope

What NA do I need to resolve these DNA strains?





3 µm

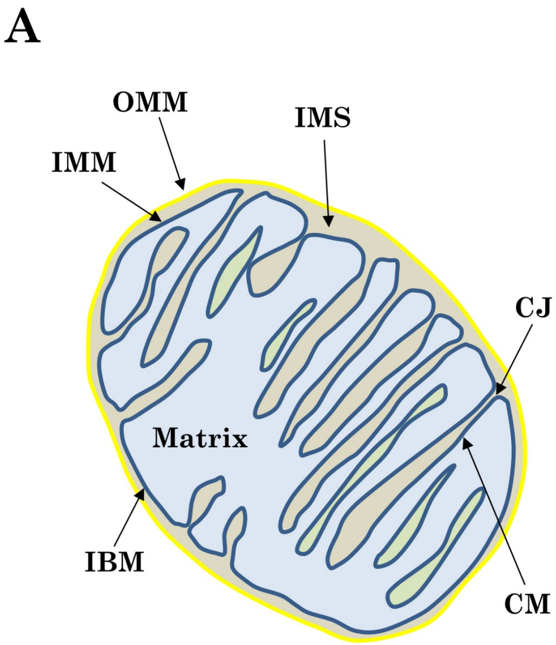


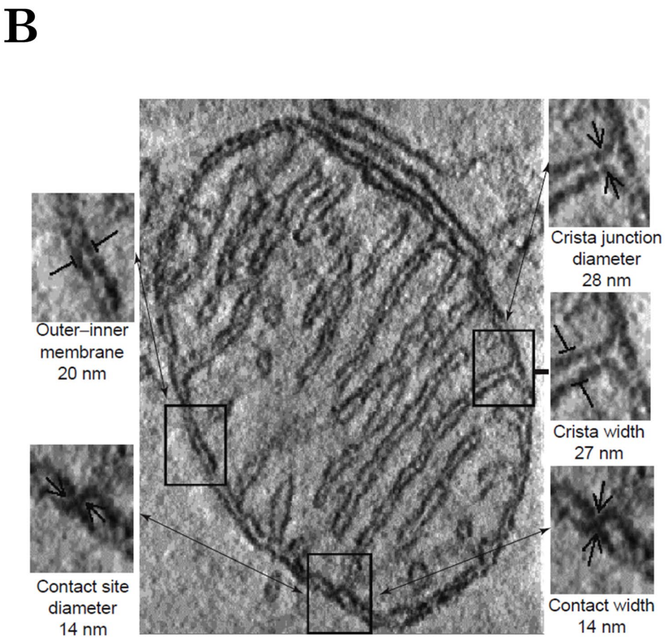
How small should my pixel size be to ensure no info is lost?

Onion root cells viewed through a light microscope. The DNA is stained purple, which allows us to see cells in various stages of cell division. Image credit: “Mitosis” by Josef Reischig, CC BY-SA 3.0. https://www.khanacademy.org/science/hs-bio/x230b3ff252126bb6:from-cells-to-organisms/x230b3ff252126bb6:the-cellular-basis-of-life/a/cell-biology-and-microscopy

Mithocondria crista

What NA do I need to resolve them?





https://www.mdpi.com/2077-0375/11/7/465

Discrete Fourier transform (DFT)

Discrete Fourier transform (DFT)

DFT “converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the reciprocal-space Fourier transform”

DFT is not an approximation of the FT

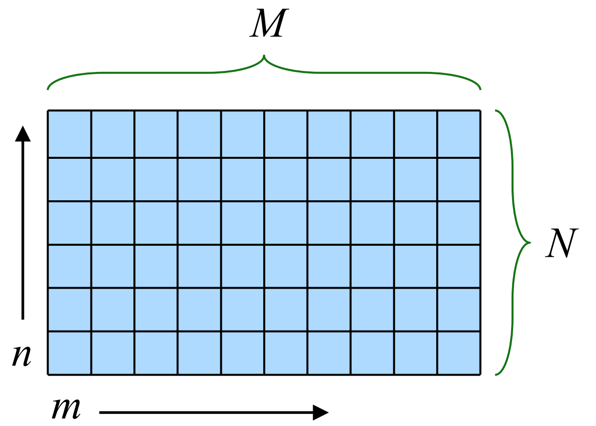
- → Exact numerical transformation
- → Orthogonal basis
- → Unique, reversible
- → Own set of theorems

Symmetric definition → Different than Goodman’s

https://en.wikipedia.org/wiki/Discrete_Fourier_transform

$$G_{p,q} = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g_{m,n} \exp\left[-i2\pi\left(\frac{mp}{M}+\frac{nq}{N}\right)\right]$$

$$g_{m,n} = \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} G_{p,q} \exp\left[i2\pi\left(\frac{mp}{M}+\frac{nq}{N}\right)\right]$$



Example DFT theorems

Shift theorems in real and reciprocal space

$$\mathrm{DFT}\left\{g_{m-l,n}\right\} = G_{p,q}\exp\left[-i2\pi\frac{lp}{M}\right]$$

$$\mathrm{DFT}\left\{g_{m,n}\exp\left[i2\pi\frac{lm}{M}\right]\right\} = G_{p-l,q}$$

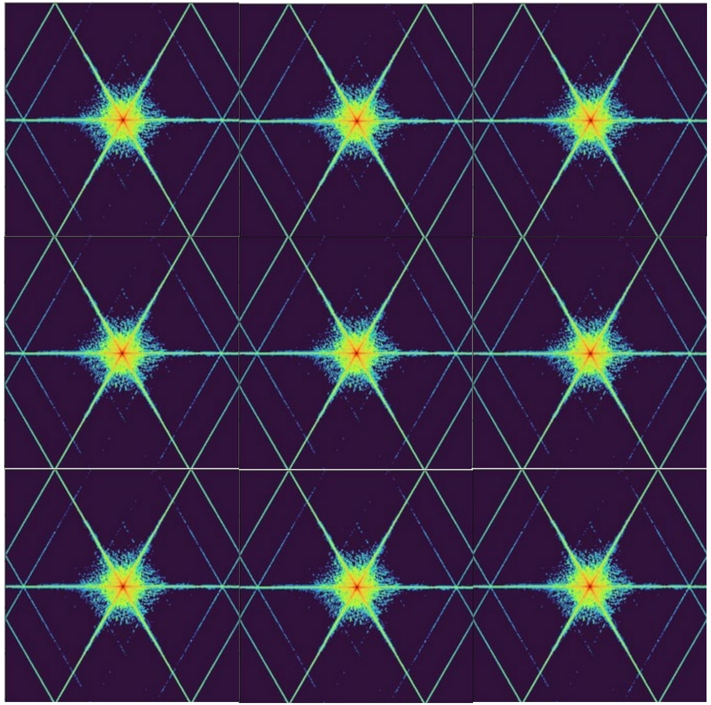
Periodic DFT

DFT views both the time and frequency domains as inherently periodic

$$\mathrm{DFT}\left\{g_{m,n}\exp\left[i2\pi\frac{lm}{M}\right]\right\}=G_{p-l,q}$$

Notice that for a shift of *M*

$$G_{p-M,q}=\mathrm{DFT}\left\{g_{m,n}\exp\left[i2\pi\frac{lM}{M}\right]\right\}=\mathrm{DFT}\left\{g_{m,n}\right\}=G_{p,q}$$



https://www.tandfonline.com/doi/full/10.1080/00268976.2024.2388303#d1e1239

Periodic DFT

DFT views both the time and frequency domains as inherently periodic

Notice that for a shift of *M*

$$G_{p-M,q} = \mathrm{DFT}\left\{ g_{m,n} \exp\left[i2\pi \frac{lM}{M} \right] \right\} = \mathrm{DFT}\left\{ g_{m,n} \right\} = G_{p,q}$$

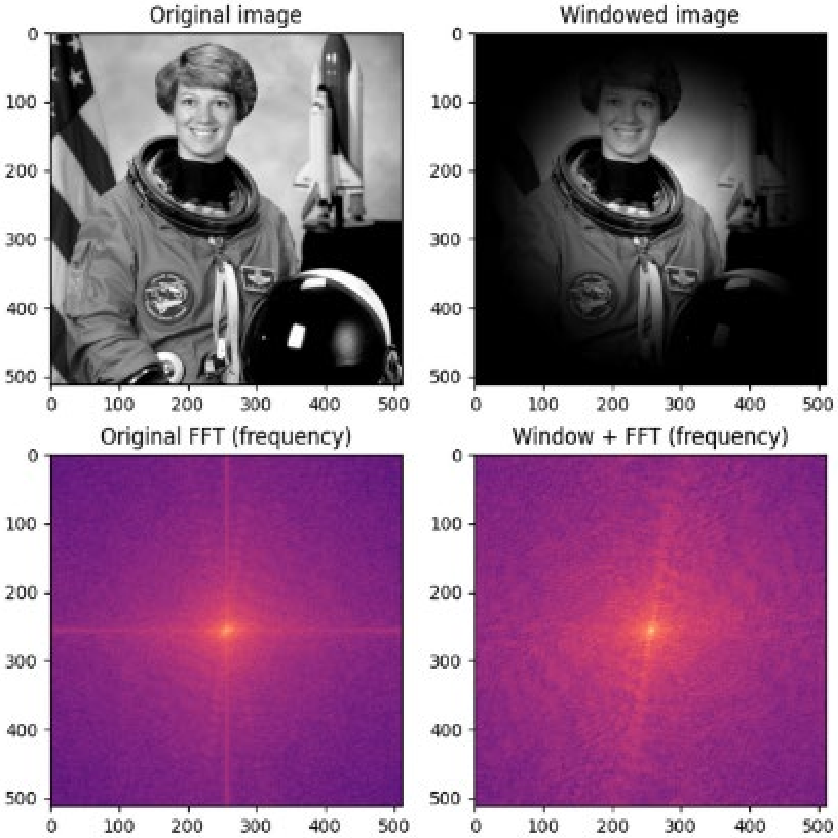
DFT will see a “sharp” edge here →

Shutterstock.com 166321793

Apodization of edges

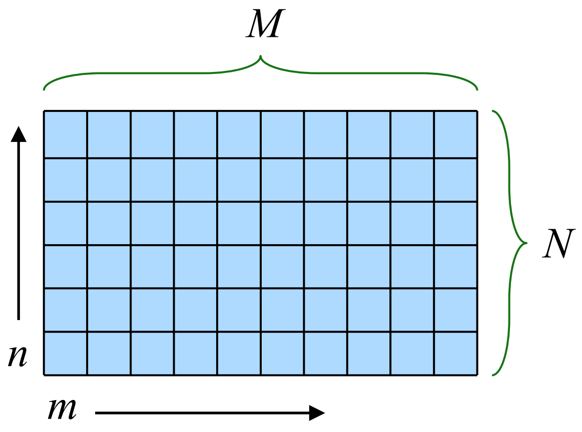
To reduce the edge effect, and get a good estimate of the frequencies really present in the image we should use image apodization

Here a Hanning window is used, but can also be a partial Hanning window.



https://scikit-image.org/docs/0.25.x/auto_examples/filters/plot_window.html

Relation to the continuous FT



Often, we do want to use the DFT to approximate the FT (notation and derivation different than Goodman's)

$$G\left(f_x, f_y\right) = \int\int_{-\infty}^{\infty} g\left(x, y\right) \exp\left(-i2\pi\left[f_x x + f_y y\right]\right) dxdy$$

$x = m\Delta x$

$y = n\Delta y$

$f_x = p\Delta f_x$

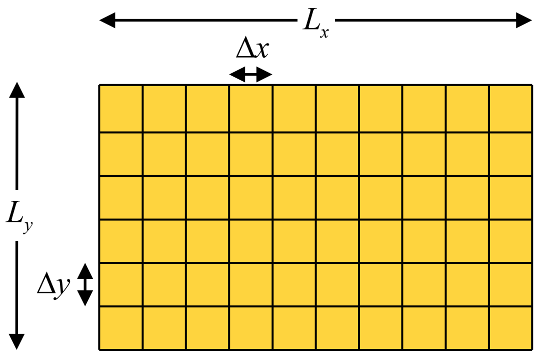
$f_y = q\Delta f_y$

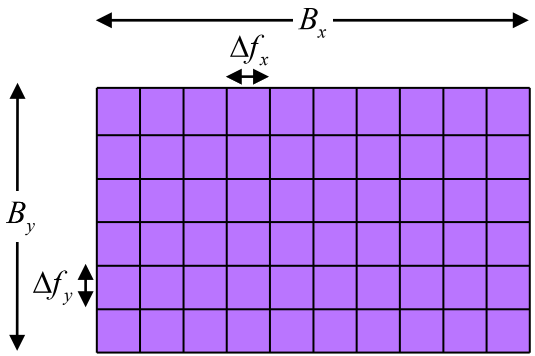
$L_x = M\Delta x$

$L_y = N\Delta y$

$B_x = M\Delta f_x$

$B_y = N\Delta f_y$





Relation to the continuous FT

$$G\left(f_x,f_y\right)=\int\int_{-\infty}^{\infty} g\left(x,y\right)\exp\left(-i2\pi\left[f_x x+f_y y\right]\right)dxdy$$

$$G\left(p\Delta f_x,q\Delta f_y\right)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g\left(m\Delta x,n\Delta y\right)\exp\left(-i2\pi\left[mp\boxed{\Delta f_x\Delta x}+nq\boxed{\Delta f_y\Delta y}\right]\right)\Delta x\Delta y$$

$$G\left(p\Delta f_x,q\Delta f_y\right)=\Delta x\Delta y\sqrt{MN}\,\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g\left(m\Delta x,n\Delta y\right)\exp\left(-i2\pi\left[\frac{mp}{\boxed{M}}+\frac{nq}{\boxed{N}}\right]\right)$$

$$G_{p,q}=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g_{m,n}\exp\left[-i2\pi\left(\frac{mp}{M}+\frac{nq}{N}\right)\right]$$

$$x=m\Delta x$$
$$y=n\Delta y$$
$$f_x=p\Delta f_x$$
$$f_y=q\Delta f_y$$

$$\Delta x\Delta f_x=\frac{1}{M}$$
$$\Delta y\Delta f_y=\frac{1}{N}$$

Relation to the continuous FT

$$G\left(f_x,f_y\right)=\int\int_{-\infty}^{\infty} g\left(x,y\right)\exp\left(-i2\pi\left[f_x x+f_y y\right]\right)dxdy$$

$$G\left(p\Delta f_x,q\Delta f_y\right)=\Delta x\Delta y\sqrt{MN}\,\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g\left(m\Delta x,n\Delta y\right)\exp\left(-i2\pi\left[\frac{mp}{M}+\frac{nq}{N}\right]\right)$$

$$G\left(p\Delta f_x,q\Delta f_y\right)=\Delta x\Delta y\sqrt{MN}\,\mathrm{DFT}\left\{g\left(m\Delta x,n\Delta y\right)\right\}$$

$x = m\Delta x$

$y = n\Delta y$

$f_x = p\Delta f_x$

$f_y = q\Delta f_y$

$$\Delta x\Delta f_x = \frac{1}{M}$$

$$\Delta y\Delta f_y = \frac{1}{N}$$

DFT on rectangular images

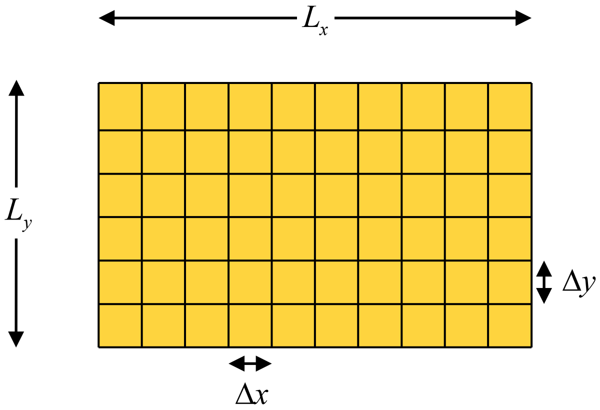
$$x = m\Delta x \qquad L_x = M\Delta x$$

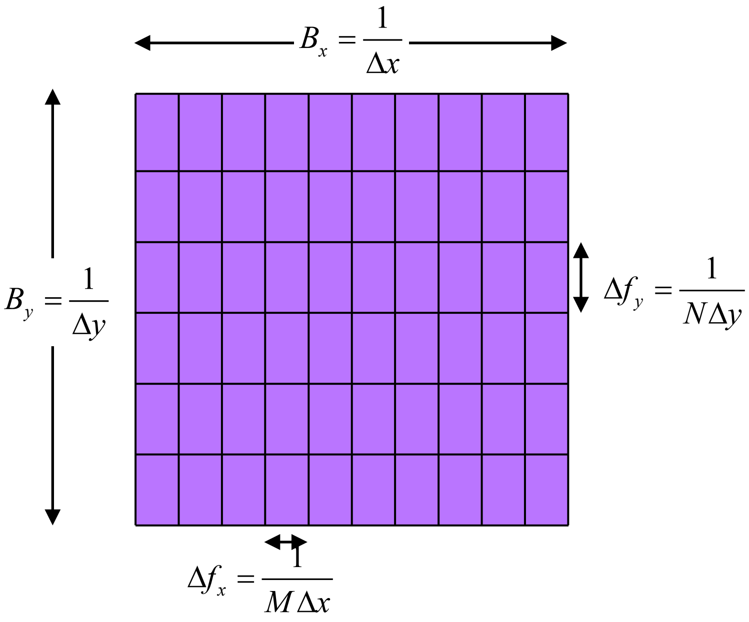
$$y = n\Delta y \qquad L_y = N\Delta y$$

$$f_x = p\Delta f_x \qquad B_x = M\Delta f_x$$

$$f_y = q\Delta f_y \qquad B_y = N\Delta f_y$$

A very common mistake in filtering rectangular images with square pixels M > N





$$\Delta x \Delta f_x = \frac{1}{M}$$

$$\Delta y \Delta f_y = \frac{1}{N}$$